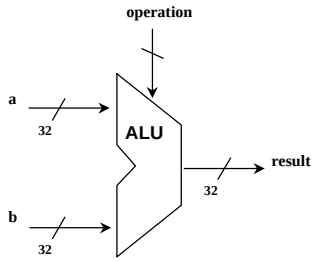
Arithmetic

- Where we've been:
  - Performance (seconds, cycles, instructions)
  - Abstractions:
    - Instruction Set Architecture
    - Assembly Language and Machine Language
- What's up ahead:
  - Implementing the Architecture



Numbers

- Bits are just bits (no inherent meaning)
  - — conventions define relationship between bits and numbers
- Binary numbers (base 2)
  - 0000 0001 0010 0011 0100 0101 0110 0111 1000 1001...
  - decimal: $0...2^n-1$
- Of course it gets more complicated:
  - numbers are finite (overflow)
  - fractions and real numbers
  - negative numbers
  - e.g., no MIPS subi instruction; addi can add a negative number)
- How do we represent negative numbers?
  - i.e., which bit patterns will represent which numbers?

Possible Representations

Sign Magnitude:	One's Complement	Two's Complement
000 = +0	000 = +0	000 = +0
001 = +1	001 = +1	001 = +1
010 = +2	010 = +2	010 = +2
011 = +3	011 = +3	011 = +3
100 = -0	100 = -3	100 = -4
101 = -1	101 = -2	101 = -3
110 = -2	110 = -1	110 = -2
111 = -3	111 = -0	111 = -1

- **Issues: balance, number of zeros, ease of operations**
- **Which one is best? Why?**

MIPS

- **32 bit signed numbers:**

$0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000_{two} = 0_{ten}$
$0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0001_{two} = +1_{ten}$
$0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0010_{two} = +2_{ten}$
...
$0111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1110_{two} = +2{,}147{,}483{,}646_{ten}$
$0111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111_{two} = +2{,}147{,}483{,}647_{ten}$ ← *maxint*
$1000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000_{two} = -2{,}147{,}483{,}648_{ten}$
$1000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0001_{two} = -2{,}147{,}483{,}647_{ten}$ ← *minint*
$1000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0000\ 0010_{two} = -2{,}147{,}483{,}646_{ten}$
...
$1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1101_{two} = -3_{ten}$
$1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1110_{two} = -2_{ten}$
$1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111_{two} = -1_{ten}$

Two's Complement Operations

- Negating a two's complement number: invert all bits and add 1
  - remember: "negate" and "invert" are quite different!
- Converting n bit numbers into numbers with more than n bits:
  - MIPS 16 bit immediate gets converted to 32 bits for arithmetic
  - copy the most significant bit (the sign bit) into the other bits

```
0010  -> 0000 0010
1010  -> 1111 1010
```

  - "sign extension" (lbu vs. lb)

Addition & Subtraction

- Just like in grade school (carry/borrow 1s)

```
  0111        0111        0110
+ 0110      - 0110      - 0101
```

- Two's complement operations easy
  - subtraction using addition of negative numbers

```
  0111
+ 1010
```

- Overflow (result too large for finite computer word):
  - e.g., adding two n-bit numbers does not yield an n-bit number

```
  0111
+ 0001
  1000
```

note that overflow term is somewhat misleading, it does not mean a carry "overflowed"

Detecting Overflow

- No overflow when adding a positive and a negative number
- No overflow when signs are the same for subtraction
- Overflow occurs when the value affects the sign:
  - overflow when adding two positives yields a negative
  - or, adding two negatives gives a positive
  - or, subtract a negative from a positive and get a negative
  - or, subtract a positive from a negative and get a positive
- Consider the operations A + B, and A – B
  - Can overflow occur if B is 0 ?
  - Can overflow occur if A is 0 ?

Effects of Overflow

- An exception (interrupt) occurs
  - Control jumps to predefined address for exception
  - Interrupted address is saved for possible resumption
- Details based on software system / language
  - example: flight control vs. homework assignment
- Don't always want to detect overflow
  — new MIPS instructions: `addu, addiu, subu`

  note: `addiu` *still sign-extends!*
  note: `sltu, sltiu` *for unsigned comparisons*

Review: Boolean Algebra & Gates

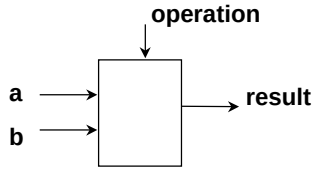
- **Problem: Consider a logic function with three inputs: A, B, and C.**

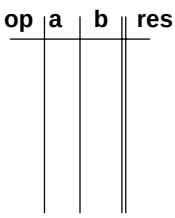
  Output D is true if at least one input is true
  Output E is true if exactly two inputs are true
  Output F is true only if all three inputs are true

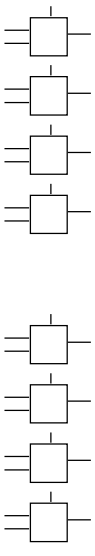
- **Show the truth table for these three functions.**
- **Show the Boolean equations for these three functions.**
- **Show an implementation consisting of inverters, AND, and OR gates.**

An ALU (arithmetic logic unit)

- **Let's build an ALU to support the `andi` and `ori` instructions**
  - **we'll just build a 1 bit ALU, and use 32 of them**



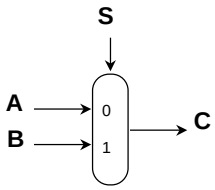




- **Possible Implementation (sum-of-products):**

Review: The Multiplexor

- **Selects one of the inputs to be the output, based on a control input**

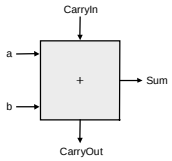


note: we call this a 2-input mux even though it has 3 inputs!

- **Lets build our ALU using a MUX:**

Different Implementations

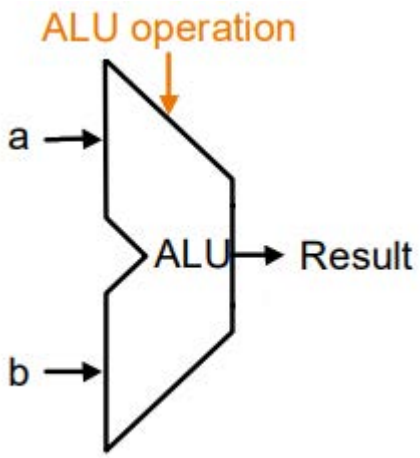
- **Not easy to decide the "best" way to build something**
  - **Don't want too many inputs to a single gate**
  - **Dont want to have to go through too many gates**
  - **for our purposes, ease of comprehension is important**
- **Let's look at a 1-bit ALU for addition:**



$c_{out} = a\ b + a\ c_{in} + b\ c_{in}$
$sum = a \text{ xor } b \text{ xor } c_{in}$

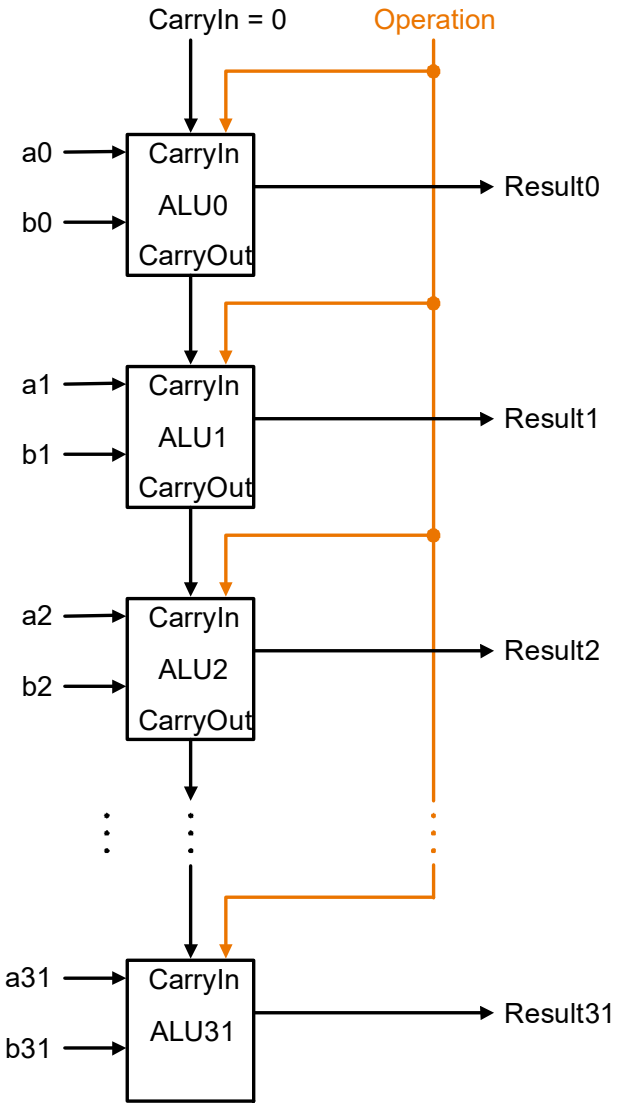
- **How could we build a 1-bit ALU for add, and, and or?**
- **How could we build a 32-bit ALU?**

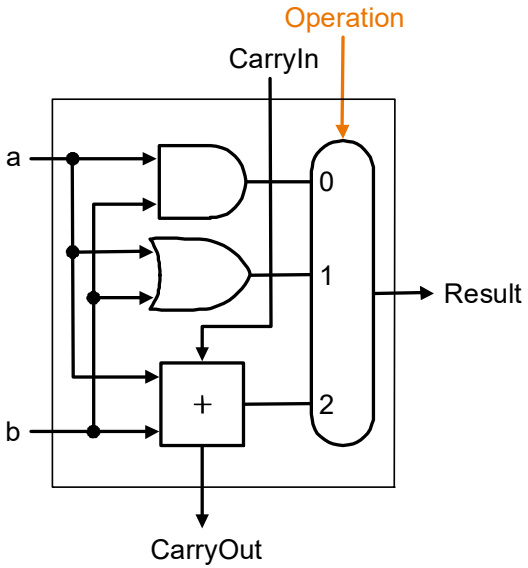
Arithmetic Logic Unit (ALU)



ALU operation (2-bit):

```
00 = and
01 = or
10 = add
```

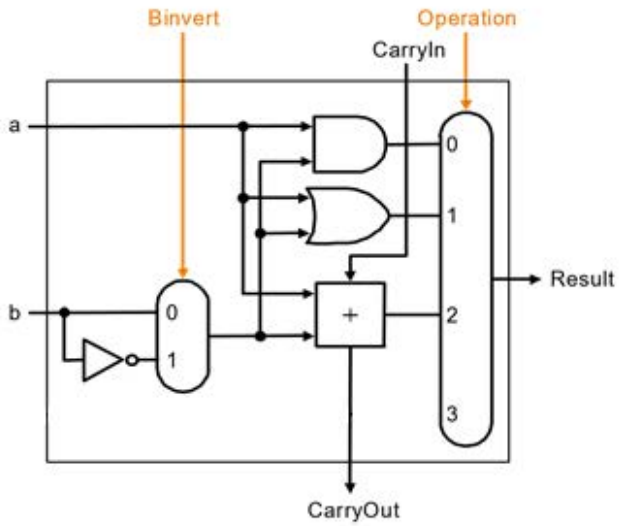




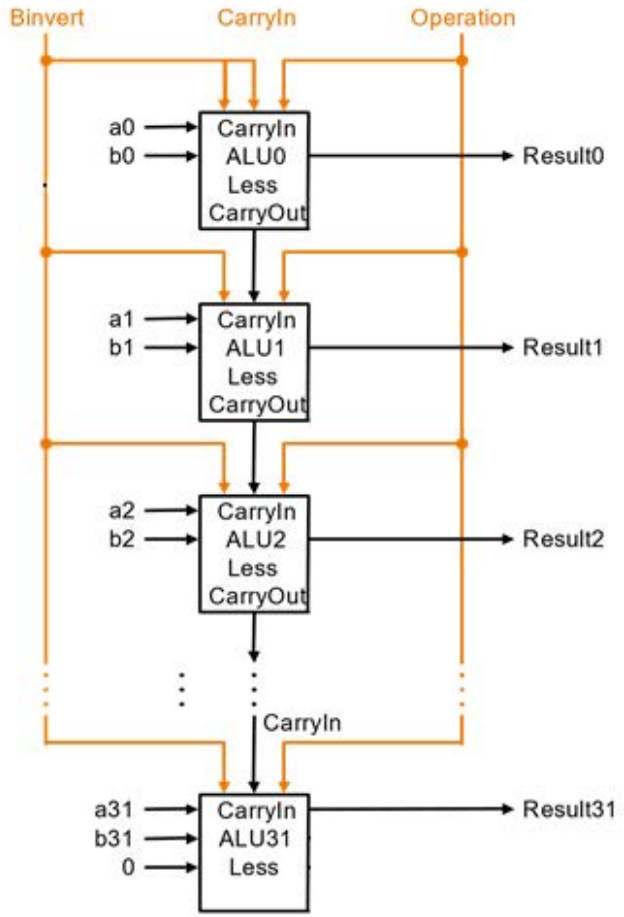
What about subtraction (a – b) ?

- Two's complement approach: just negate b and add.
- How do we negate?

Invert all bits of b



Add 1



ALU operation (3-bit):

```
Binvert Operation
   0        00        = and
   0        01        = or
   0        10        = add
   1        10        = sub
```