# Estimating from Outputs of Oversampled Delta-Sigma Modulation*

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**Abstract.** Oversampling a delta-sigma-modulated sequence, one can compute unbiased sample estimates of averages of consecutive input elements for a wide variety of inputs. We prove that these estimates are most efficient in their class (that is, variances of sample means are minimum in the class of random binary sequences $g_n$, $n = 1, \ldots, N$, such that the expected values of $g_n$ are equal to the values of the corresponding inputs of delta-sigma modulation) and consistent. Delta-sigma modulation may also be described as one-dimensional error diffusion (a technique for digital halftoning). However, delta-sigma modulation is not a practical digital halftoning algorithm, because human vision averages small luminance deviations in two dimensions. We pose an open problem that invites the reader to extend our approach to the two-dimensional case for the purpose of development of a practical digital halftoning algorithm.

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## 1. Introduction

*Delta-sigma (or sigma-delta) modulation* [3, 20] is a well-known form of data transformation applied in digital signal processing and communication systems. Our paper deals with single-loop delta-sigma modulation (more sophisticated configurations are known [4, 13]). Without loss of generality, we take the range of delta-sigma modulation to be $[0, 1]$ (linear transformations cover arbitrary ranges $[\gamma_1, \gamma_2]$, $\gamma_1 < \gamma_2$). We also assume that the values of inputs $x_n$, $n = 1, 2, \ldots, N$ lie within $[0, 1]$, i.e. there is no *overload*. This is a common and reasonable assumption [7]. Under our assumptions, the delta-sigma modulation procedure guarantees that the outputs of the binary quantizer $q_n$, $n = 1, \ldots, N$, can be computed from the inputs $x_n$, $n = 1, \ldots, N$, by the formula

$$q_n = \frac{1}{2} - \frac{1}{2}\mathrm{sgn}'\left(2x_n - 1 + 2\sum_{i=1}^{n-1}(x_i - q_i)\right), \tag{1}$$

where values of function $\mathrm{sgn}'()$ are computed as

$$\mathrm{sgn}'(z) = \begin{cases} 1 & \text{if } z \geq 0, \\ -1 & \text{if } z < 0. \end{cases} \tag{2}$$

Moreover, $q_n \in [0, 1]$ for all $n$. Eq. (1) is equivalent to

$$q_n = \begin{cases} 1 & \text{if } x_n + \sum\limits_{i=1}^{n-1}(x_i - q_i) \geq \dfrac{1}{2}, \\ 0 & \text{if } x_n + \sum\limits_{i=1}^{n-1}(x_i - q_i) < \dfrac{1}{2}. \end{cases} \tag{3}$$

Let $s_{n-1} = \sum_{i=1}^{n-1}(x_i - q_i) + \frac{1}{2}$. Then

$$s_n = x_n - q_n + s_{n-1}. \tag{4}$$

Let $\lfloor \mu \rfloor$ denote the *integer part* of $\mu$ and $\langle \mu \rangle = (\mu \bmod 1)$ its *fractional part*. Then ranges of $x_n$ and $q_n$ allow Eq. (3) to be rewritten as

$$q_n = \lfloor x_n + s_{n-1} \rfloor. \tag{5}$$

From the fact that $\mu - \langle \mu \rangle = \lfloor \mu \rfloor$ we get

$$x_n - q_n = \langle x_n + s_{n-1} \rangle - s_{n-1}. \tag{6}$$

Plugging the expression from the right-hand side of Eq. (6) into Eq. (4) yields

$$s_n = \langle x_n + s_{n-1} \rangle. \tag{7}$$

Let $x_0 = s_0 = 1/2$. Using formula (7) and taking into account that $\langle \mu + \langle \chi \rangle \rangle = \langle \mu + \chi \rangle$, we transform Eq. (5) and get

$$\begin{aligned} q_n &= \lfloor x_n + s_{n-1} \rfloor = \lfloor x_n + \langle x_{n-1} + s_{n-2} \rangle \rfloor \\ &= \lfloor x_n + \langle x_{n-1} + \langle x_{n-2} + s_{n-3} \rangle \rangle \rfloor = \lfloor x_n + \langle x_{n-1} + x_{n-2} + s_{n-3} \rangle \rfloor = \cdots \\ &= \left\lfloor x_n + \left\langle \sum_{i=1}^{n-1} x_i + s_0 \right\rangle \right\rfloor = \left\lfloor x_n + \left\langle \sum_{i=0}^{n-1} x_i \right\rangle \right\rfloor . \end{aligned} \tag{8}$$

Oversampled delta-sigma modulators introduced by Inose and Yasuda [16] can be used to compute sample means [9]

$$u_M = \frac{1}{N}\sum_{i=0}^{N-1} q_{M-i}, \tag{9}$$

where $M > N$.

Candy [2] observed that values $u_M$ represent the average value of the input signal and derived an $O(1/N)$ bound on the error. Most of the extensive research of properties of sequences related to delta-sigma modulation involved statistical and spectral analysis of the *quantization noise sequence* $\epsilon_n = \sum_{i=1}^{n}(q_i - x_i)$ and the *quantization error sequence* $e_n = q_n - x_n$ for different inputs [7, 9, 10, 11, 12, 23]. Our paper, on the other hand, deals mainly with properties of outputs of oversampled delta-sigma modulation as estimates of averages of consecutive input elements.

The input sequence $x_n$, $n = 1, 2, \dots, N$, can be either random, or deterministic. Galton [7] randomized $x_n$ by introducing an additive i.i.d. noise component into the input. For the purposes of our analysis, we randomize the coding procedure by means of introducing a random parameter $\alpha$ uniformly distributed on $[0, 1]$ into Eq. (8):

$$q_n = \left\lfloor x_n + \left\langle \sum_{i=1}^{n-1} x_i + \alpha \right\rangle \right\rfloor . \tag{10}$$

Parameter $\alpha$ can be interpreted as a random initial value $x_0$ (or $s_0$) of the sequence encoded. Besides purely technical reasons for introduction of $\alpha$, we take into account the following considerations.

If sequence $x_n$, $n = 1, 2, \dots, N$, is a random one, then random variable $\langle X_n \rangle$, where $X_n = \sum_{i=1}^{n} x_i$, can be treated as the initial value for calculation of sequence $q_{n+i}$, $i = 1, 2, \dots, N - n$, and has distribution close to uniform under very relaxed conditions ([5], pp. 62–63). (Sequence $q_{n+i}$, in its turn, can be used to estimate averages by computing sample means.)

If $x_n$, $n = 1, 2, \dots, N$, is a deterministic sequence, then $\langle X_n \rangle$, $n = 1, 2, \dots, N$, is uniformly distributed [22] in a sufficiently general case (i.e., frequency of the values $\langle X_n \rangle$ being in any range $(a, b) \subset [0, 1]$ goes to $(b - a)$ when $N$ goes to infinity), so the delta-sigma modulation model with a random initial value applies to many deterministic input sequences as well.

Our paper proves that the (unbiased) estimates of averages of consecutive input elements computed according to Eq. (9) are most efficient in their class (that is, variances of sample means are minimum in the class of random binary sequences $g_n$, $n = 1, \dots, N$, such that the expected values of $g_n$ are equal to the values of the corresponding inputs of delta-sigma modulation) for inputs that allow application of our delta-sigma modulation model with a random initial value. We also give a simple proof of consistency of these estimates, a property that can be easily derived using the results from [2], [9], or [7].

Anastassiou [1] observed that delta-sigma modulation can be interpreted as one-dimensional error diffusion (a digital halftoning technique applied in image processing). Hein and Zakhor [14] used this relation for the purposes of halftone to continuous-tone conversion of error-diffusion coded images. However, delta-sigma modulation itself is not a practical halftoning technique, because human vision averages over small luminance deviations in two dimensions. As we shall see, the estimates being most efficient in their class explains the orientation of the characteristic correlated artifacts known to affect the performance of line-by-line and column-by-column delta-sigma modulation as a halftoning technique. Error diffusion is known for similar artifacts, and Sandler et al. [19] used some related considerations to justify the selection of an error diffusion algorithm they combined with other digital halftoning techniques. In the end of the article, we will pose an open problem inviting the reader to use our results to design a practical digital halftoning algorithm.

## 2. Probabilistic Model

We introduce the following probabilistic model of delta-sigma modulation equivalent (as we will show) to the one described by Eq. (10). Let $O$ be a circle with length 1, $I_1 \subset O$ an arbitrary interval with its length $|I_1|$ equal to $x_1$, and $\beta$ a random variable distributed uniformly on [0,1].

Let

$$q_1 = \begin{cases} 0 & \text{if } \beta \notin I_1, \\ 1 & \text{if } \beta \in I_1. \end{cases} \tag{11}$$

Clearly, the expected value and variance of $q_1$ are $E(q_1) = \text{Prob}(q_1 = 1) = x_1$ and $V(q_1) = x_1(1 - x_1)$, respectively.

Random variable $q_2$ can now be produced using interval $I_2 \in O$ such that $|I_2| = x_2$. Then $E(q_2) = x_2$, $V(q_2) = x_2(1 - x_2)$, and $V(q_1 + q_2) = V(q_1) + V(q_2) + 2\rho(q_1, q_2)$, where $\rho(\mu, \chi)$ stands for the correlation coefficient of random variables $\mu$ and $\chi$. The expected value of random variable $Q_2 = (q_1 + q_2)/2$ is $(x_1 + x_2)/2$, and the variance of this variable is minimal when $\rho(q_1, q_2)$ is minimal. But $\rho(q_1, q_2) = \text{Prob}(q_1 = 1, q_2 = 1) - x_1 x_2$, so it is smaller when the measure (overall length) of $I_1 \cap I_2$ is smaller. Let's introduce counterclockwise orientation on $O$. Now each of the intervals $I_1$ and $I_2$ has its beginning and end. Moreover, if the beginning of interval $I_2$ coincides with the end of $I_1$, then the measure of their intersection is minimal. Let's place $I_2$ on the circle this way, thus completing our derivation of $q_2$. To conclude the description of the probabilistic model, it is sufficient to mention that intervals $I_3, \ldots, I_N$ will be treated similarly. We apply this model below to prove related statistical properties of delta-sigma-modulated sequences.

## 3. Proof of Statistical Properties

First, notice that

$$\rho(q_1, q_2) = \begin{cases} -x_1 x_2 & \text{if } x_1 + x_2 < 1, \\ x_1 + x_2 - 1 - x_1 x_2 & \text{if } x_1 + x_2 \geq 1. \end{cases} \tag{12}$$

Eq. (12) corresponds to the minimum correlation coefficient [22] of two binary random variables, one of which takes value 1 with probability $x_1$, and the other one does so with probability $x_2$. Hence, whenever it is required that $E(q_1) = x_1$ and $E(q_2) = x_2$, our way to construct $q_1$ and $q_2$ minimizes variance of $Q_2$.

Let's produce random variable $q_3$ analogously. Namely, by placing interval $I_3$, $|I_3| = x_3$, so that its beginning coincides with the end of interval $I_2$. The expected value of random variable $Q_3 = (q_1 + q_2 + q_3)/3$ is equal to $(x_1 + x_2 + x_3)/3$. We are about to show that, under the condition that $E(q_3) = x_3$, variance of this new random variable is also minimum.

First, we demonstrate that no other placement of interval $I_3$ can decrease $V(Q_3)$. Consider random variable

$$c = \begin{cases} 0 & \text{if } \beta \notin I_1 \cup I_2, \\ 1 & \text{if } \beta \in (I_1 \cup I_2) \backslash (I_1 \cap I_2), \\ 2 & \text{if } \beta \in I_1 \cap I_2. \end{cases} \tag{13}$$

Clearly, $c = q_1 + q_2$ and

$$V(q_1 + q_2 + q_3) = V(c + q_3) = V(c) + V(q_3) + 2\rho(c, q_3), \tag{14}$$

where $V(c)$ is minimum by construction and $V(q_3) = x_3(1 - x_3)$.

We are yet to show that our placement of $I_3$ minimizes the correlation coefficient

$$\rho(c, q_3) = E(cq_3) - (x_1 + x_2)x_3, \tag{15}$$

i.e., the minimum of $E(cq_3)$ is achieved.

When $x_1 + x_2 \leq 1$, the result follows from the reasoning we used when constructing random variable $q_2$. We simply substitute $I_1 \cup I_2$ for $I_1$. When $x_1 + x_2 > 1$,

$$E(cq_3) = 2 \cdot |I_1 \cap I_2 \cap I_3| + 1 \cdot (|I_3| - |I_1 \cap I_2 \cap I_3|) = |I_1 \cap I_2 \cap I_3| + |I_1|, \tag{16}$$

and this is minimum when the beginning of interval $I_3$ coincides with the end of interval $I_1 \cup I_2$, that is, with the end of $I_2$.

Continuation of this procedure leads to the following construction. Using the beginning of interval $I_1$ as "zero", place intervals $I_2, I_3, \dots, I_N$ so that the beginning of $I_{n+1}$ coincides with the end of $I_n$, $n = 1, 2, \dots, N$. Then form values

$$q_n = \begin{cases} 0 & \text{if } \beta \notin I_n, \\ 1 & \text{if } \beta \in I_n. \end{cases} \tag{17}$$

To show that sequences determined by equations (10) and (17) coincide, we consider $O$ as the result of reduction modulo 1, make a change of variables $x' = x - \beta$, and drop the prime. Then the beginning of interval $I_1$ will get shifted clockwise by $\beta$ and Eq. (17) will become

$$q_n = \begin{cases} 0 & \text{if } 0 \notin I_n, \\ 1 & \text{if } 0 \in I_n. \end{cases} \tag{18}$$

Finally, we replace random variable $\beta$ with $\alpha = 1 - \beta$, which is also uniformly distributed on $[0, 1]$.

Thus, random variable

$$Q_N = \frac{1}{N} \sum_{n=1}^{N} q_n, \tag{19}$$

where $q_n$, $n = 1, 2, \dots, N$, are determined by Eq. (10), has expected value

$$E(Q_N) = \frac{1}{N} \sum_{n=1}^{N} x_n, \tag{20}$$

and, by simple induction, the variance of this random variable for all $N > 1$ is minimum in the class of random binary sequences $g_n$, $n = 1, \dots, N$, such that

$$E(g_n) = x_n. \tag{21}$$

Hence, if demodulation on the $M$th step is performed by computing $u_M$, an estimate of the average $(1/N) \sum_{i=0}^{N-1} x_{M-i}$, according to Eq. (9), then $u_M$ is *most efficient* in the class of estimates computed as

$$v_M = \frac{1}{N} \sum_{i=0}^{N-1} g_{M-i}, \tag{22}$$

where $g_n$ is a random binary sequence satisfying Eq. (21), when values $q_n$ are determined by formula (10).

It is important to note that this property is true for any input sequence which allows application of our probabilistic model. In particular, if that sequence is random, then the property is true for all of its realizations. Condition (21) is then equivalent to the requirement that estimates $v_M$ are *unbiased* ($u_M$ being unbiased follows immediately from [7 (Theorem 3)]).

Let's show that sample means of consecutive outputs of the binary quantizer approach averages of corresponding elements of sequence $x_n$ when size of the sample increases. Indeed, from the construction above it follows that

$$\sum_{n=1}^{N} q_n = \left\lfloor \sum_{n=1}^{N} x_n + \alpha \right\rfloor, \tag{23}$$

and inequality

$$\left| \alpha + \sum_{n=1}^{N} x_n - \sum_{n=1}^{N} q_n \right| < 1 \tag{24}$$

is true for all $\alpha \in [0,1]$. Therefore,

$$\left| \frac{1}{N} \sum_{n=1}^{N} x_n - \frac{1}{N} \sum_{n=1}^{N} q_n \right| < \frac{1}{N}, \quad N = 1, 2, \ldots. \tag{25}$$

Considering what happens to Ineq. (25) when $N$ goes to 0, we find that delta-sigma modulation guarantees *consistency* of sample means of consecutive outputs of the binary quantizer as estimates of averages of corresponding consecutive elements for any sequence $x_n$, $n = 1, \ldots, N$, normalized so that $x_n \in [0,1]$ for all $n$. Our having randomized the coding procedure did not actually matter here, so the consistency property is present regardless of the nature of the input sequence. Note that this property is absent when common nearest-level quantization is applied instead. It is straightforward to show consistency using the results of Candy [2 (Eq. (5))], Gray [9 (Corollary 1)], or Galton [7 (Corollary 6)].

Recall that $x_n \in [0,1]$ and $q_n \in [0,1]$ for all $n$. From (5) and (7) it follows that

$$q_n = \begin{cases} 1 & \text{if } x_n + s_{n-1} \geq 1, \\ 0 & \text{if } x_n + s_{n-1} < 1. \end{cases} \tag{26}$$

Equations (7) and (10), our probabilistic model, and results from [5, 22] imply that values of $s_{n-1}$ are often uniformly distributed on $[0,1)$. When this is the case, one more interesting fact can be derived.

Eq. (4) implies that *quantization error* for any element of sequence $x_n$ is

$$e_n = q_n - x_n = s_{n-1} - s_n, \tag{27}$$

and

$$\varepsilon_j = \sum_{i=1}^{j} (q_i - x_i) = \sum_{i=1}^{j} (s_{i-1} - s_i) = s_0 - s_j \tag{28}$$

is the value of *quantization noise* after $j$ elements of sequence $x_n$ are processed. Notice that, since $s_n$ are uniformly distributed on $[0,1)$ and $s_0 \in [0,1)$ is a constant, the expected value of $|\varepsilon_n|$ is

$$E(|\varepsilon_n|) = 1/2 - s_0 + s_0^2. \tag{29}$$

Hence the expected absolute value of the quantization noise is minimal (and equal to $1/4$) when $s_0 = 1/2$.

## 4. Open Problem

Human vision possesses ability ([18], p. 621) spatially to average small luminance deviations. (Related neurobiological aspects of vision are discussed in [15].) Error diffusion [6] and other algorithms of digital halftoning (see, for example, [8, 17, 21]) take advantage of this averaging process, which can be loosely described as two-dimensional "demodulation" of graphic information. Moreover, appropriate choices of error diffusion coefficients lead to line-by-line and column-by-column delta-sigma modulation of images [1]. The sample mean of the consecutive outputs of the binary quantizer being the most efficient estimate of the average of the corresponding inputs in its class explains why line-by-line delta-sigma modulation of images produces characteristic corellated artifacts (zebra patterns) oriented vertically. (Analogously, column-by-column delta-sigma modulation produces horizontally oriented zebra stripes.) Error diffusion is known for similar corellated artifacts (orientation may vary), and a related discussion based on the results of spectral analysis can be found in [1]. Describe two-dimensional analogs of statistical criteria met (in one dimension) by delta-sigma modulation. Design a digital halftoning algorithm meeting these new criteria and compare it to existing algorithms.

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