Fundamentals of Probability

Instructor: Dmitri A. Gusev

Fall 2007

CS 502: Computers and Communications Technology

Lecture 16, October 31, 2007

Reading

- http://mathworld.wolfram.com/Probability.html

Probability

- *Probability* is the chance that a particular event will occur expressed on a linear scale from 0 (impossibility) to 1 (certainty), or as a percentage between 0 and 100%.
- Before the middle of the 17th century, the term 'probable' (Latin *probabilis*) meant ‘approvable’. A probable action or opinion was one of the kind that sensible people would choose to take or hold.

Frequency Approach

- Under the frequency approach, probability is measured as the frequency of outcomes of independent, well-defined (sufficiently similarly conducted) independent random experiments, such as flipping a coin or rolling a die. Probability of the event E is

  $P(E) = N(E)/N$,

  where N is the number of experiments and N(E) is the number of experiments with the outcome E.

Bayesian Approach

- Probability exists as a quantitative expression of the subjective plausibility that the event will occur, even if we cannot estimate its value by measurement.
- We can estimate parameters of an underlying distribution based on the properties of the observed distribution (a histogram).

Probability Axioms

Let

$$S \equiv \bigcup_{i=1}^{N} E_i ,$$

where N is either finite or ∞, and let $P(E_i)$ be the probability of the event E_i. Then

1. $0 \le P(E_i) \le 1$.
2. $P(S)=1$.
3. Additivity: $P(E_i U E_j) = P(E_i)+P(E_j)$ if E_i and E_j are mutually exclusive. In particular, $P(E'_i) = 1 – P(E_i)$.
4. Countable additivity:

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

for $n=1,2,3,\ldots,N$, where $E_1,E_2,\ldots,E_N$ are mutually exclusive.

Joint Probability

- $P(A \cdot B) = P(A) \cdot P(B)$ if *A* and *B* are independent

Conditional Probability

- The conditional probability of an event assuming that has occurred equals

$$P(A|B)=\frac{P(A\cdot B)}{P(B)}$$

Bayes Theorem

$$P(A|B) = \frac{P(A) \cdot P(B|A)}{P(B)}$$

Interpretation: A is the cause, B is the effect. If B occurred, what is the probability that the cause of that was A?

Law of Total Probability

$$P(B) = \sum_{i=1}^{n} P(A_i)P(B \mid A_i)$$

where $A_1, \ldots, A_n$ are mutually exclusive and exhaustive (i.e., *B* can occur with one and only one of them).

Bayesian Formula (Laplas)

$$P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{\sum_{j=1}^{n} P(A_j)P(B \mid A_j)}$$

Interpretation: Event *B* may occur under one and only one of Conditions A_1, A_2, …, A_n. Their apriori probabilities are $P(A_i)$. An experiment is conducted, in which Event *B* did occur. *Aposteriori* probabilities of the conditions are different from the *apriori* ones. In other words, we re-evaluate the probabilities of these conditions based on the outcome of our experiment.

Example

- 5 urns with white and black balls:
  - 2 urns of type A_1: 2 white and 3 black balls
  - 2 urns of type A_2: 1 white and 4 black balls
  - 1 urn of type A_3: 4 white and 1 black balls

A ball was taken without looking from a randomly picked urn. The ball is white. What is the probability of its having come from an urn of type A_3?