On First-Order Two-Dimensional Linear Homogeneous Partial Difference Equations*

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Abstract

Analysis of algorithms occasionally requires solving of first-order two-dimensional linear homogeneous partial difference equations. We survey solutions to special cases of the linear recurrence equation $(am + bn + c)F_{m,n} = (dm + en + f)F_{m-1,n} + (gm + hn + i)F_{m-1,n-1}$ in terms of known functions and establish equivalences between unsolved cases. The article also reviews solution techniques used to simplify recurrences and establish equivalences between them.

1 Introduction

The standard form for the recurrence equation we will be considering is

$$x(m,n)F_{m,n} = y(m,n)F_{m-1,n} + z(m,n)F_{m-1,n-1}.$$

Recurrences of this form arise in problems related to analysis of algorithms and combinatorics (see, for example, [7], Sections 6.3.1 and 8.2).

We are especially interested in the case when $x(m,n) \equiv am + bn + c$, $y(m,n) \equiv dm + en + f$, $z(m,n) \equiv gm + hn + i$, i. e.,

$$(am + bn + c)F_{m,n} = (dm + en + f)F_{m-1,n} + (gm + hn + i)F_{m-1,n-1},$$

where a, b, c, d, e, f, g, h, and i are real numbers. Our goal is to determine which forms of this recurrence have solutions in terms of known functions. If we classify these forms according to which of the coefficients a, b, d, e, g, and h are non-zero, there are 64 distinct cases to consider. The values of the constant terms c, f, and i have little effect on whether a given form has a solution. Redefining variables or shifting indices will usually eliminate them from the recurrence.

Tables of solutions in Section 3 contain particular solutions that satisfy the corresponding equations. Finding solutions that also fit particular boundary conditions is a separate problem. It is discussed in [6] and related to a well-known similar problem for partial differential equations considered in [2, 3] and many other sources. For simplicity's sake we also ignore the problem of finding the radii of convergence for the solutions we find. Interested readers are referred to [9].

2 Notation and Canonical Recurrences

In this section, we review the notation for several well-known functions and remind the reader of recurrences these functions satisfy. We also provide references to sources where one may read more about these functions (how to compute them, their properties, etc.). In fact, there are many more sources, because the functions under consideration are extremely common.

1. $F_{m,n} = F_{m-1,n} + F_{m-1,n-1}$ is satisfied by the *binomial coefficients* $\binom{m}{n}$ (see [7], Section 3.2, and [1], pp. 822–823).

*This work was partially supported by NSF Grants CCR 92-03942 and CCR 94-02780.

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2. $F_{m,n} = (m-1)F_{m-1,n} + F_{m-1,n-1}$ is satisfied by the *Stirling numbers of the first kind* $\left[{m \atop n}\right]$ ([1], p. 824).

3. $F_{m,n} = nF_{m-1,n} + F_{m-1,n-1}$ is satisfied by the *Stirling numbers of the second kind* $\left\{{m \atop n}\right\}$ ([1], pp. 824–825).

4. $F_{m,n} = nF_{m-1,n} + (m-n+1)F_{m-1,n-1}$ is satisfied by the *Eulerian numbers* $\left\langle{m \atop n}\right\rangle$ ([4], p.35).

5. $F_{m,n} = (m+n-1)F_{m-1,n} + F_{m-1,n-1}$ is satisfied by the Lah numbers $\frac{(m-1)!}{(n-1)!}\binom{m}{n}$ (see [5] and [7], p. 353).

6. *Riordan's recurrence* $F_{m,n} = F_{m-1,n} + (m+n-1)F_{m-1,n-1}$ has the solution $\frac{(m+n)!}{n!(m-n)!2^n}$ ([8], p. 85).

7. *Kummer's equation* $z\frac{d^2w}{dz^2} + (b-z)\frac{dw}{dz} - aw = 0$ has two independent solutions denoted by $M(a,b,z)$ and $U(a,b,z)$ ([1], p. 504). Recurrence $(m-1)F_{m,n} = (m-n)F_{m-1,n} + (n-1)F_{m-1,n-1}$ is satisfied by $M(m,n,z)$ for all z. Recurrence $cF_{m,n} = (m-1)F_{m-1,n} - (m-1)F_{m-1,n-1}$ is satisfied by $M(n,m,c)$. $cF_{m,n} = (m-n-1)F_{m-1,n} + F_{m-1,n-1}$ is satisfied by $U(n,m,c)$.

8. *Gauss hypergeometric function* $F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)z^n}{\Gamma(c+n)n!}$ has the following properties ([1], p. 556–558). $a(m-1)F_{m,n} = (m+n-k-1)F_{m-1,n} + (k-n)F_{m-1,n-1}$ is satisfied by $F(m,n;k;1-a)$. $a(m-k-1)F_{m,n} = (a-1)(m-1)F_{m-1,n} + (m-1)F_{m-1,n-1}$ is satisfied by $F(n,k;m;a)$.

3 Tables of Solutions

The 64 forms of the general recurrence are divided into 17 cases with solutions in terms of known functions and 47 cases with no such known solutions. The 47 cases with no known solution can be further divided into classes of cases solvable in terms of one another but not in terms of known functions. Using the techniques of substitution, permutation, and index shifting described in Section 4, it is possible to exhibit 14 such equivalence classes among the 47 unsolved cases. Table 3.1 presents general case solutions and equivalences.

The cases with no known general solution are sometimes solvable if certain relations are introduced between the values of the coefficients a, b, c, d, e, f, g, h, and i. Some special case solutions are given in Table 3.2.

4 Solution Techniques

The techniques of substitution, permutation, and index shifting were used in the preparation of the above tables with one of two purposes: first, to transform a recurrence into a form with a known solution; second, to transform an unknown recurrence into another unknown recurrence and so exhibit the equivalence between them.

4.1 Simplification by Substitution

In the case that $x(m,n) = x(m)$ we may simplify the recurrence by making the substitution $F_{m,n} = p(m)H_{m,n}$ where p is yet to be determined ([7], Section 8.2.2):

$$F_{m,n} = p(m)H_{m,n},$$

$$x(m)p(m)H_{m,n} = y(m,n)p(m-1)H_{m-1,n} + z(m,n)p(m-1)H_{m-1,n-1},$$

$$x(m)\frac{p(m)}{p(m-1)}H_{m,n} = y(m,n)H_{m-1,n} + z(m,n)H_{m-1,n-1}.$$

Table 3.1

Case	$x(m,n)$	$y(m,n)$	$z(m,n)$	Solution (or Equivalence)
0	c	f	i	$\frac{f^m i^n}{c^m f^n}\binom{m}{n}$
1	c	f	$hn + i$	$\frac{f^m h^n \Gamma(n+i/h+1)}{c^m f^n}\binom{m}{n}$
2	c	f	$gm + i$	$\frac{g^n f^m}{f^n c^m}\begin{bmatrix} m+i/g+1 \\ m-n \end{bmatrix}$
3	c	f	$gm+hn+i$	special cases
4	c	$en + f$	i	$\frac{e^m i^n}{c^m e^n}\begin{Bmatrix} m \\ n+f/e \end{Bmatrix}$
5	c	$en + f$	$hn + i$	$\frac{e^m h^n \Gamma(n+i/h+1)}{c^m e^n}\begin{Bmatrix} m \\ n+f/e \end{Bmatrix}$
6	c	$en + f$	$gm + i$	?
7	c	$en + f$	$gm+hn+i$	special case
8	c	$dm + f$	i	$\frac{d^m i^n}{c^m d^n}\begin{bmatrix} m+f/d+1 \\ n \end{bmatrix}$
9	c	$dm + f$	$hn + i$	$\frac{d^m h^n \Gamma(n+i/h+1)}{c^m d^n}\begin{bmatrix} m+f/d+1 \\ n \end{bmatrix}$
10	c	$dm + f$	$gm + i$	special case
11	c	$dm + f$	$gm+hn+i$	?
12	c	$dm+en+f$	i	Case 3
13	c	$dm+en+f$	$hn + i$	Case 3
14	c	$dm+en+f$	$gm + i$	Case 11
15	c	$dm+en+f$	$gm+hn+i$	?
16	$bn + c$	f	i	$\frac{(-i)^n f^{m+1}}{f^n b^{m+1}}\begin{Bmatrix} n-m-1 \\ n+c/b \end{Bmatrix}$
17	$bn + c$	f	$hn + i$	$\frac{f^{m+1}(-h)^n \Gamma(n+i/h+1)}{f^n b^{m+1}}\begin{Bmatrix} n-m-1 \\ n+c/b \end{Bmatrix}$
18	$bn + c$	f	$gm + i$	$\frac{f^{m+1} g^n}{b^{m+1} f^n}\left\langle \begin{matrix} n-m+c/b-i/g-2 \\ n+c/b \end{matrix} \right\rangle$
19	$bn + c$	f	$gm+hn+i$	?
20	$bn + c$	$en + f$	i	Case 10
21	$bn + c$	$en + f$	$hn + i$	Case 10
22	$bn + c$	$en + f$	$gm + i$	?
23	$bn + c$	$en + f$	$gm+hn+i$	?
24	$bn + c$	$dm + f$	i	Case 6
25	$bn + c$	$dm + f$	$hn + i$	Case 6
26	$bn + c$	$dm + f$	$gm + i$	Case 22
27	$bn + c$	$dm + f$	$gm+hn+i$	?
28	$bn + c$	$dm+en+f$	i	Case 11
29	$bn + c$	$dm+en+f$	$hn + i$	Case 11
30	$bn + c$	$dm+en+f$	$gm + i$	?
31	$bn + c$	$dm+en+f$	$gm+hn+i$	?

Table 3.1 (continued)

Case	$x(m,n)$	$y(m,n)$	$z(m,n)$	Solution (or Equivalence)
32	$am + c$	f	i	$\dfrac{f^m i^n}{a^m f^n \Gamma(m+c/a+1)} \dbinom{m}{n}$
33	$am + c$	f	$hn + i$	$\dfrac{f^m h^n \Gamma(n+i/h+1)}{a^m f^n \Gamma(m+c/a+1)} \dbinom{m}{n}$
34	$am + c$	f	$gm + i$	$\dfrac{g^n f^m}{f^n a^m \Gamma(m+c/a+1)} \begin{bmatrix} m+i/g+1 \\ m-n \end{bmatrix}$
35	$am + c$	f	$gm+hn+i$	Case 3
36	$am + c$	$en + f$	i	$\dfrac{e^m i^n}{a^m e^n \Gamma(m+c/a+1)} \begin{Bmatrix} m \\ n+f/e \end{Bmatrix}$
37	$am + c$	$en + f$	$hn + i$	$\dfrac{e^m h^n \Gamma(n+i/h+1)}{a^m e^n \Gamma(m+c/a+1)} \begin{Bmatrix} m \\ n+f/e \end{Bmatrix}$
38	$am + c$	$en + f$	$gm + i$	Case 6
39	$am + c$	$en + f$	$gm+hn+i$	Case 7
40	$am + c$	$dm + f$	i	$\dfrac{d^m i^n}{a^m d^n \Gamma(m+c/a+1)} \begin{bmatrix} m+f/d+1 \\ n \end{bmatrix}$
41	$am + c$	$dm + f$	$hn + i$	$\dfrac{d^m h^n \Gamma(n+i/h+1)}{a^m d^n \Gamma(m+c/a+1)} \begin{bmatrix} m+f/d+1 \\ n \end{bmatrix}$
42	$am + c$	$dm + f$	$gm + i$	Case 10
43	$am + c$	$dm + f$	$gm+hn+i$	Case 11
44	$am + c$	$dm+en+f$	i	Case 3
45	$am + c$	$dm+en+f$	$hn + i$	Case 3
46	$am + c$	$dm+en+f$	$gm + i$	Case 11
47	$am + c$	$dm+en+f$	$gm+hn+i$	Case 15
48	$am+bn+c$	f	i	Case 3
49	$am+bn+c$	f	$hn + i$	Case 3
50	$am+bn+c$	f	$gm + i$	Case 19
51	$am+bn+c$	f	$gm+hn+i$	Case 15
52	$am+bn+c$	$en + f$	i	Case 11
53	$am+bn+c$	$en + f$	$hn + i$	Case 11
54	$am+bn+c$	$en + f$	$gm + i$	Case 27
55	$am+bn+c$	$en + f$	$gm+hn+i$	?
56	$am+bn+c$	$dm + f$	i	Case 7
57	$am+bn+c$	$dm + f$	$hn + i$	Case 7
58	$am+bn+c$	$dm + f$	$gm + i$	Case 23
59	$am+bn+c$	$dm + f$	$gm+hn+i$	Case 55
60	$am+bn+c$	$dm+en+f$	i	Case 15
61	$am+bn+c$	$dm+en+f$	$hn + i$	Case 15
62	$am+bn+c$	$dm+en+f$	$gm + i$	Case 31
63	$am+bn+c$	$dm+en+f$	$gm+hn+i$	?

Table 3.2

Case	$x(m,n)$	$y(m,n)$	$z(m,n)$	Solution
3a	c	f	$gm + gn + i$	$\dfrac{f^m g^n (m+n+i/g+1)!}{c^m f^n (n+i/g-1)!(m-n-i/g+3)! 2^{n+i/g-1}}$
3b	c	f	$2gm - gn + i$	$\dfrac{f^m g^n m!}{c^m f^n (m-n+i/g+1)!} \dbinom{m-1}{m-n+i/g}$
7a	c	$en + f$	$gm - gn + i$	$\dfrac{g^n e^{m+1}}{e^n c^{m+1}} \left\langle {m+f/e+i/g-1 \atop n+f/e} \right\rangle$
10a	c	$dm + f$	$k(dm + f)$	$\dfrac{d^m k^n \Gamma(m+f/d+1)}{c^m} \dbinom{m}{n}$

Solving the secondary recurrence for p

$$x(m)p(m) = p(m-1)$$

results in the simplification

$$H_{m,n} = y(m,n)H_{m-1,n} + z(m,n)H_{m-1,n-1}.$$

Likewise, in the case that $z(m,n) = z(n)$ we may simplify the recurrence by making the substitution $F_{m,n} = q(n)H_{m,n}$ (where q is yet to be determined):

$$F_{m,n} = q(n)H_{m,n},$$

$$x(m,n)q(n)H_{m,n} = y(m,n)q(n)H_{m-1,n} + z(n)q(n-1)H_{m-1,n-1},$$

$$x(m,n)H_{m,n} = y(m,n)H_{m-1,n} + z(n)\frac{q(n-1)}{q(n)}H_{m-1,n-1}.$$

Solving the secondary recurrence for q

$$q(n) = z(n)q(n-1)$$

results in the simplification

$$x(m,n)H_{m,n} = y(m,n)H_{m-1,n} + H_{m-1,n-1}.$$

Case 0 from Table 3.1 is a good example of a simple recurrence that can be solved with this technique.

4.2 Simplification by Shifting Indices

Standard techniques may be used to shift the indices of the recurrence equation. This may result in simplification, as shown in the following example. Solve

$$F_{m,n} = (n-4)F_{m-1,n} + F_{m-1,n-1}.$$

Define

$$G_{m,n-4} \stackrel{\text{def}}{=} F_{m,n}.$$

Then

$$G_{m,n-4} = (n-4)G_{m-1,n-4} + G_{m-1,n-5},$$

$$G_{m,n} = nG_{m-1,n} + G_{m-1,n-1}.$$

This is the familiar recurrence for Stirling numbers of the second kind. Hence,

$$G_{m,n} = \left\{ {m \atop n} \right\},$$

$$F_{m,n} = \left\{ {m \atop n-4} \right\}.$$

4.3 Simplification by Permutation

The general recurrence

$$x(m,n)H_{m-\alpha,n-\beta} = y(m,n)H_{m-\gamma,n-\delta} + z(m,n)H_{m-\epsilon,n-\zeta}$$

may be transformed into a recurrence in the standard form

$$x'(m',n')F_{m',n'} = y'(m',n')F_{m'-1,n'} + z'(m',n')F_{m'-1,n'-1}$$

by an appropriate coordinate translation and rotation.

First, translate the indices

$$H_{m-\alpha,n-\beta} \equiv G_{m,n}$$

to yield

$$x(m,n)G_{m,n} = y(m,n)G_{m+\alpha-\gamma,n+\beta-\delta} + z(m,n)G_{m+\alpha-\epsilon,n+\beta-\zeta}.$$

Second, following [7], rotate the coordinates by replacing m and n in x, y, and z according to

$$\Delta \stackrel{\text{def}}{=} (\gamma-\alpha)(\zeta-\beta) - (\delta-\beta)(\epsilon-\alpha),$$

$$x'(m',n') = x((\gamma-\alpha)m' - (\gamma-\epsilon)n', (\delta-\beta)m' + (\zeta-\delta)n'),$$

$$y'(m',n') = x((\gamma-\alpha)m' - (\gamma-\epsilon)n', (\delta-\beta)m' + (\zeta-\delta)n'),$$

$$z'(m',n') = x((\gamma-\alpha)m' - (\gamma-\epsilon)n', (\delta-\beta)m' + (\zeta-\delta)n').$$

(If $\Delta = 0$ the terms of the recurrence are colinear and the recurrence may be solved as a one-dimensional recurrence.) This rotation yields a new recurrence in standard form in terms of the variables m' and n'. If this new recurrence can be solved, then the solution to the original recurrence can be written down by means of backward substitution for m' and n' according to

$$G_{m,n} = F_{\frac{(\zeta-\delta)m+(\gamma-\epsilon)n}{\Delta},\frac{(\beta-\delta)m+(\gamma-\alpha)n}{\Delta}},$$

$$H_{m,n} = G_{m+\alpha,n+\beta} = F_{\frac{(\zeta-\delta)(m+\alpha)+(\gamma-\epsilon)(n+\beta)}{\Delta},\frac{(\beta-\delta)(m+\alpha)+(\gamma-\alpha)(n+\beta)}{\Delta}}.$$

Making use of the above technique we may permute the order of the coefficients in this recurrence according to the following table:

Table 4.1

	Forward Substitution					Backward Substitution	
Permutation	$/x$	$/y$	$/z$	$/m$	$/n$	$/m'$	$/n'$
(1)(2)(3)	x	y	z	m'	n'	m	n
(12)(3)	y	x	$-z$	$n'-m'$	n'	$n-m-1$	n
(13)(2)	z	$-y$	x	$-n'$	$-m'$	$-n-1$	$-m-1$
(1)(23)	x	z	y	m'	$m'-n'$	m	$m-n$
(123)	z	x	$-y$	$n'-m'$	$-m'$	$-n-1$	$m-n$
(132)	y	$-z$	x	$-n'$	$m'-n'$	$n-m-1$	$-m-1$

Here the forward substitutions are used to transform one recurrence in standard form into another recurrence in standard form in terms of the variables m' and n'. This second recurrence may then be solved. Backward substitution into this solution for m' and n' in terms of the original variables m and n yields a solution to the original recurrence.

To illustrate the effect of permutation (123) on the general recurrence

$$x(m,n)H_{m,n} = y(m,n)H_{m-1,n} + z(m,n)H_{m-1,n-1},$$

we may permute according to the forward substitutions to yield:

$$z(n'-m',-m')F_{m',n'} = x(n'-m',-m')F_{m'-1,n'} - y(n'-m',m')F_{m'-1,n'-1}.$$

The solution to the original recurrence is then obtained from the solution to this recurrence by backward substitution for m' and n'

$$H_{m,n} = F_{-n-1,m-n}.$$

To see the permutation technique in action, consider the following example. Solve

$$nH_{m,n} = H_{m-1,n} + (m-1)H_{m-1,n-1}.$$

Permute according to (12)(3) to yield

$$F_{m',n'} = n'F_{m'-1,n'} - (n'-m'-1)F_{m'-1,n'-1},$$

$$F_{m',n'} = n'F_{m'-1,n'} + (m'-n'+1)F_{m'-1,n'-1}.$$

This is the familiar Eulerian recurrence, which leads to the solution

$$F_{m',n'} = \left\langle {m' \atop n'} \right\rangle.$$

Backward substitution yields the solution of the original recurrence

$$F_{m,n} = \left\langle {n-m-1 \atop n} \right\rangle.$$

Acknowledgment. The authors thank Paul W. Purdom, Jr. for his helpful suggestions on presentation of the paper.

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